

# Representing graphs as the intersection of axis-parallel cubes

(Extended Abstract)

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**Abstract.** A unit cube in  $k$  dimensional space (or  $k$ -cube in short) is defined as the Cartesian product  $R_1 \times R_2 \times \cdots \times R_k$  where  $R_i$  (for  $1 \leq i \leq k$ ) is a closed interval of the form  $[a_i, a_i + 1]$  on the real line. A  $k$ -cube representation of a graph  $G$  is a mapping of the vertices of  $G$  to  $k$ -cubes such that two vertices in  $G$  are adjacent if and only if their corresponding  $k$ -cubes have a non-empty intersection. The *cubicity* of  $G$ , denoted as  $\text{cub}(G)$ , is the minimum  $k$  such that  $G$  has a  $k$ -cube representation. Roberts [17] showed that for any graph  $G$  on  $n$  vertices,  $\text{cub}(G) \leq 2n/3$ . Many NP-complete graph problems have polynomial time deterministic algorithms or have good approximation ratios in graphs of low cubicity. In most of these algorithms, computing a low dimensional cube representation of the given graph is usually the first step.

From a geometric embedding point of view, a  $k$ -cube representation of  $G = (V, E)$  yields an embedding  $f : V \rightarrow \mathbb{R}^k$  such that for any two vertices  $u$  and  $v$ ,  $\|f(u) - f(v)\|_\infty \leq 1$  if and only if  $(u, v) \in E$ .

We present an efficient algorithm to compute the  $k$ -cube representation of  $G$  with maximum degree  $\Delta$  in  $O(\Delta \ln n)$  dimensions. We then further strengthen this bound by giving an algorithm that produces a  $k$ -cube representation of a given graph  $G$  with maximum degree  $\Delta$  in  $O(\Delta \ln b)$  dimensions where  $b$  is the bandwidth of  $G$ . Bandwidth of  $G$  is at most  $n$  and can be much lower. The algorithm takes as input a bandwidth ordering of the vertices in  $G$ . Though computing the bandwidth ordering of vertices for a graph is NP-hard, there are heuristics that perform very well in practice. Even theoretically, there is an  $O(\log^4 n)$  approximation algorithm for computing the bandwidth ordering of a graph using which our algorithm can produce a  $k$ -cube representation of any given graph in  $k = O(\Delta(\ln b + \ln \ln n))$  dimensions. Both the bounds on cubicity are shown to be tight upto a factor of  $O(\log \log n)$ .

**Keywords:** Cubicity, bandwidth, intersection graphs, unit interval graphs.

## 1 Introduction

Let  $\mathcal{F} = \{S_x \subseteq U : x \in V\}$  be a family of subsets of a universe  $U$ , where  $V$  is an index set. The intersection graph  $\Omega(\mathcal{F})$  of  $\mathcal{F}$  has  $V$  as vertex set, and two distinct

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vertices  $x$  and  $y$  are adjacent if and only if  $S_x \cap S_y \neq \emptyset$ . Representations of graphs as the intersection graphs of various geometrical objects is a well studied topic in graph theory. Probably the most well studied class of intersection graphs are the *interval graphs*, where each  $S_x$  is a closed interval on the real line. A restricted form of interval graphs, that allow only intervals of unit length, are *indifference graphs*.

A well known concept in this area of graph theory is the *cubicity*, which was introduced by F. S. Roberts in 1969 [17]. This concept generalizes the concept of indifference graphs. A unit cube in  $k$  dimensions ( $k$ -cube) is a Cartesian product  $R_1 \times R_2 \times \cdots \times R_k$  where  $R_i$  (for  $1 \leq i \leq k$ ) is a closed interval of the form  $[a_i, a_i + 1]$  on the real line. Two  $k$ -cubes,  $(x_1, x_2, \dots, x_k)$  and  $(y_1, y_2, \dots, y_k)$  are said to have a non- empty intersection if and only if the intervals  $x_i$  and  $y_i$  have a non-empty intersection for  $1 \leq i \leq k$ . For a graph  $G$ , its *cubicity* is the minimum dimension  $k$ , such that  $G$  is representable as the intersection graph of  $k$ -cubes. We denote the cubicity of a graph  $G$  by  $\text{cub}(G)$ . Note that a  $k$ -cube representation of  $G$  using cubes with unit side length is equivalent to a  $k$ -cube representation where the cubes have side length  $c$  for some fixed positive constant  $c$ . The graphs of cubicity 1 are exactly the class of indifference graphs. The cubicity of a complete graph is taken to be 0. If we require that each vertex correspond to a  $k$ -dimensional axis-parallel box  $R_1 \times R_2 \times \cdots \times R_k$  where  $R_i$  (for  $1 \leq i \leq k$ ) is a closed interval of the form  $[a_i, b_i]$  on the real line, then the minimum dimension required to represent  $G$  is called its *boxicity* denoted as  $\text{box}(G)$ . Clearly  $\text{box}(G) \leq \text{cub}(G)$  for any graph  $G$  because cubicity is a stricter notion than boxicity. It has been shown that deciding whether the cubicity of a given graph is at least 3 is NP-hard [22]. As for boxicity, it was shown by Kratochvil [13] that deciding whether the boxicity of a graph is at most 2 is NP-complete.

In many algorithmic problems related to graphs, the availability of certain convenient representations turn out to be extremely useful. Probably, the most well-known and important examples are the tree decompositions and path decompositions. Many NP-hard problems are known to be polynomial time solvable given a tree(path) decomposition of the input graph that has bounded width. Similarly, the representation of graphs as intersections of “disks” or “spheres” lies at the core of solving problems related to frequency assignments in radio networks, computing molecular conformations etc. For the maximum independent set problem which is hard to approximate within a factor of  $n^{(1/2)-\epsilon}$  for general graphs, a PTAS is known for disk graphs given the disk representation [9, 1]. In a similar way, the availability of cube or box representation in low dimension makes some well known NP-hard problems like the max-clique problem, polynomial time solvable since there are only  $O((2n)^k)$  maximal cliques if the boxicity or cubicity is at most  $k$ . Though the maximum independent set problem is hard to approximate within a factor  $n^{(1/2)-\epsilon}$  for general graphs, it is approximable to a  $\log n$  factor for boxicity 2 graphs (the problem is NP-hard even for boxicity 2 graphs) given a box or cube representation [2, 3].

It is easy to see that the problem of representing graphs using  $k$ -cubes can be equivalently formulated as the following geometric embedding problem. Given an undirected unweighted graph  $G = (V, E)$  and a threshold  $t$ , find an embedding  $f : V \rightarrow \mathbb{R}^k$  of the vertices of  $G$  into a  $k$ -dimensional space (for the minimum possible  $k$ ) such that for any two vertices  $u$  and  $v$  of  $G$ ,  $\|f(u) - f(v)\|_\infty \leq t$  if and only if  $u$  and  $v$  are adjacent. The norm  $\| \cdot \|_\infty$  is the  $L_\infty$  norm. Clearly, a  $k$ -cube representation of  $G$  yields the required embedding of  $G$  in the  $k$ -dimensional space. The minimum dimension required to embed  $G$  as above under the  $L_2$  norm is called the *sphericity* of  $G$ . Refer to [15] for applications where such an embedding under  $L_\infty$  norm is argued to be more appropriate than embedding under  $L_2$  norm. The connection between cubicity and sphericity of graphs were studied in [11, 14].

Roberts [17] showed that for any graph  $G$  on  $n$  vertices,  $\text{cub}(G) \leq 2n/3$ . The cube representation of special class of graphs like hypercubes and complete multipartite graphs were investigated in [17, 14, 16]. Similarly, the boxicity of special classes of graphs were studied by [18, 19, 8]. An algorithm to compute the box representation in  $O(\Delta \ln n)$  dimensions for any graph  $G$  on  $n$  vertices and maximum degree  $\Delta$  was shown in [4]. Researchers have also tried to generalize or extend the concept of boxicity in various ways. The poset boxicity, the rectangle number, grid dimension, circular dimension and the boxicity of digraphs are some examples.

Two recent results about the boxicity of any graph  $G$  on  $n$  vertices and having maximum degree  $\Delta$  are  $\text{box}(G) = O(\Delta \ln n)$  [4] and  $\text{box}(G) \leq 2\Delta^2$  [5]. Combining these with the result  $\frac{\text{cub}(G)}{\text{box}(G)} \leq \lceil \log n \rceil$  shown in [7], we get  $\text{cub}(G) = O(\Delta \ln^2 n)$  and  $\text{cub}(G) \leq 2\Delta^2 \lceil \log n \rceil$ . Our result is an improvement over both these bounds on cubicity.

*Linear arrangement and Bandwidth.* Given an undirected graph  $G = (V, E)$  on  $n$  vertices, a *linear arrangement* of the vertices of  $G$  is a bijection  $L : V \rightarrow \{1, \dots, n\}$ . The *width* of the linear arrangement  $L$  is defined as  $\max_{(u, v) \in E} |L(u) - L(v)|$ . The *bandwidth minimization problem* is to compute  $L$  with minimum possible width. The *bandwidth* of  $G$  denoted as  $b$  is the minimum possible width achieved by any linear arrangement of  $G$ . A *bandwidth ordering* of  $G$  is a linear arrangement of  $V(G)$  with width  $b$ .

## 1.1 Our results

We summarize below the results of this paper.

Let  $G$  be a graph on  $n$  vertices. Let  $\Delta$  be the maximum degree of  $G$  and  $b$  its bandwidth.

We first show a randomized algorithm to construct the cube representation of  $G$  in  $O(\Delta \ln n)$  dimensions. This randomized construction can be easily derandomized to obtain a polynomial time deterministic algorithm that gives a cube representation of  $G$  in the same number of dimensions. We then give a second algorithm that takes as input a linear arrangement of the vertices of  $G$  with width  $b$  to construct the  $k$ -cube representation of  $G$  in  $k = O(\Delta \ln b)$  dimensions.

Note that the bandwidth  $b$  is at most  $n$  and  $b$  is much smaller than  $n$  for many well-known graph classes.

Note that the second algorithm to compute the cube representation of a graph  $G$  takes as input a linear arrangement of  $V(G)$ . The smaller the width of this arrangement, the lesser the number of dimensions of the cube representation of  $G$  computed by our algorithm. It is NP-hard to approximate the bandwidth of  $G$  within a ratio better than  $k$  for every fixed  $k \in \mathbb{N}$  [21]. Feige [10] gives a  $O(\log^3(n)\sqrt{\log n \log \log n})$  approximation algorithm to compute the bandwidth (and also the corresponding linear arrangement) of general graphs using which we obtain polynomial time deterministic or randomized algorithms to construct the cube representation of  $G$  in  $O(\Delta(\ln b + \ln \ln n))$  dimensions, given only  $G$ . It should be noted that several algorithms with good heuristics that perform very well in practice [20] are known for bandwidth computation.

We also show that the bounds on cubicity given by both our algorithms are tight up to a factor of  $O(\log \log n)$ .

## 1.2 Definitions and Notations

Let  $G$  be a simple, finite, undirected graph on  $n$  vertices. The vertex set of  $G$  is denoted as  $V(G) = \{1, \dots, n\}$  (or  $V$  in short). Let  $E(G)$  (or  $E$  in short) denote the edge set of  $G$ . Let  $G'$  be a graph such that  $V(G') = V(G)$ . Then  $G'$  is a *supergraph* of  $G$  if  $E(G) \subseteq E(G')$ . We define the *intersection* of two graphs as follows. If  $G_1$  and  $G_2$  are two graphs such that  $V(G_1) = V(G_2)$ , then the intersection of  $G_1$  and  $G_2$  denoted as  $G = G_1 \cap G_2$  is graph with  $V(G) = V(G_1) = V(G_2)$  and  $E(G) = E(G_1) \cap E(G_2)$ . For a vertex  $u \in V(G)$ ,  $N(u)$  denotes the set of neighbours of  $u$ . The degree of the vertex  $u$  in  $G$  is denoted by  $d(u)$  and  $d(u) = |N(u)|$ . Let  $\Delta$  denote the maximum degree of  $G$ . Let  $b$  denote the bandwidth of  $G$ .

## 1.3 Indifference graph representation

Let  $G = (V, E)$  be a graph and  $I_1, \dots, I_k$  be  $k$  indifference graphs such that  $V(I_i) = V(G)$  and  $E(G) \subseteq E(I_i)$ , for  $1 \leq i \leq k$ . If  $G = I_1 \cap \dots \cap I_k$ , then we say that  $I_1, \dots, I_k$  is an *indifference graph representation* of  $G$ . The following theorem due to Roberts relates  $\text{cub}(G)$  to the indifference graph representation of  $G$ .

**Theorem 1.** *A graph  $G$  has  $\text{cub}(G) \leq k$  if and only if it can be expressed as the intersection of  $k$  indifference graphs.*

All our algorithms compute an indifference graph representation of  $G$ . It is straightforward to derive the cube representation of  $G$  given its indifference graph representation. To describe an indifference graph, we define a function  $f : V \rightarrow \mathbb{R}$  such that for a vertex  $u$ , its closed interval is given by  $[f(u), f(u) + l]$ , for a fixed constant  $l$  which is assumed to be 1 unless otherwise specified. Note that even if the  $l$  value is different for each of the component indifference graphs, the unit cube representation can be derived by scaling down all the intervals of each component indifference graph by the corresponding  $l$  value.

## 2 Cube representation in $O(\Delta \ln n)$ dimensions

In this section we describe an algorithm to compute the cube representation of any graph  $G$  on  $n$  vertices and maximum degree  $\Delta$  in  $O(\Delta \ln n)$  dimensions.

**Definition 1.** Let  $\pi$  be a permutation of the set  $\{1, \dots, n\}$ . Let  $X \subseteq \{1, \dots, n\}$ . The projection of  $\pi$  onto  $X$  denoted as  $\pi_X$  is defined as follows. Let  $X = \{u_1, \dots, u_r\}$  such that  $\pi(u_1) < \pi(u_2) < \dots < \pi(u_r)$ . Then  $\pi_X(u_1) = 1, \pi_X(u_2) = 2, \dots, \pi_X(u_r) = r$ .

**Construction of indifference supergraph given  $\pi$ :**

Let  $G(V, E)$  be a simple, undirected graph. Let  $\pi$  be a permutation on  $V$  and let  $A$  be a subset of  $V$ . We define  $\mathcal{M}(G, \pi, A)$  to be an indifference graph  $G'$  constructed as follows:

Let  $B = V - A$ . We now assign intervals of length  $n$  to the vertices in  $V$ . The function  $f$  defines the left end-points of the intervals (of length  $n$ ) mapped to each vertex as follows:

$$\begin{aligned} \forall u \in B, \text{ define } f(u) &= n + \pi(u), \\ \forall u \in A \text{ and } N(u) \cap B = \emptyset, \text{ define } f(u) &= 0, \\ \forall u \in A \text{ and } N(u) \cap B \neq \emptyset, \text{ define } f(u) &= \max_{x \in N(u) \cap B} \{\pi(x)\}. \end{aligned}$$

$G'$  is the intersection graph of these intervals. Thus, two vertices  $u$  and  $v$  will have an edge in  $G'$  if and only if  $|f(u) - f(v)| \leq n$ . Since each vertex is mapped to an interval of length  $n$ ,  $G'$  is an indifference graph. It can be seen that the vertices in  $B$  induce a clique in  $G'$  as the intervals assigned to each of them contain the point  $2n$ . Similarly, all the vertices in  $A$  also induce a clique in  $G'$  as the intervals mapped to each contain the point  $n$ .

Now, we show that  $G'$  is a supergraph of  $G$ . To see this, take any edge  $(u, v) \in E(G)$ . If  $u$  and  $v$  both belong to  $A$  or if both belong to  $B$ , then  $(u, v) \in E(G')$  as we have observed above. If this is not the case, then we can assume without loss of generality that  $u \in A$  and  $v \in B$ . Let  $t = \max_{x \in N(u) \cap B} \{\pi(x)\}$ . Obviously,  $t \geq \pi(v)$ , since  $v \in N(u) \cap B$ . From the definition of  $f$ , we have  $f(u) = t$  and we have  $f(v) = n + \pi(v)$ . Therefore,  $f(v) - f(u) = n + \pi(v) - t$  and since  $t \geq \pi(v)$ , it follows that  $f(v) - f(u) \leq n$ . This shows that  $(u, v) \in E(G')$ .

We give a randomized algorithm **RAND** that, given an input graph  $G$ , outputs an indifference supergraph  $G'$  of  $G$ .

**RAND**

Input:  $G$ .

Output:  $G'$  which is an indifference supergraph of  $G$ .

**begin**

- Step 1. Generate a permutation  $\pi$  of  $\{1, \dots, n\}$  uniformly at random.
- Step 2. For each vertex  $u \in V$ , toss an unbiased coin to decide whether it should belong to  $A$  or to  $B$  (i.e.  $\Pr[u \in A] = \Pr[u \in B] = \frac{1}{2}$ ).
- Step 3. Return  $G' = \mathcal{M}(G, \pi, A)$ .

**end**

**Lemma 1.** Let  $e = (u, v) \notin E(G)$ . Let  $G'$  be the graph returned by **RAND**( $G$ ). Then,

$$\begin{aligned}\Pr[e \in E(G')] &\leq \frac{1}{2} + \frac{1}{4} \left( \frac{d(u)}{d(u) + 1} + \frac{d(v)}{d(v) + 1} \right) \\ &\leq \frac{2\Delta + 1}{2\Delta + 2}\end{aligned}$$

where  $\Delta$  is the maximum degree of  $G$ .

*Proof.* Let  $\pi$  be the permutation and  $\{A, B\}$  be the partition of  $V$  generated randomly by **RAND**( $G$ ). An edge  $e = (u, v) \notin E(G)$  will be present in  $G'$  if and only if one of the following cases occur:

1. Both  $u, v \in A$  or both  $u, v \in B$
2.  $u \in A, v \in B$  and  $\max_{x \in N(u) \cap B} \pi(x) > \pi(v)$
3.  $u \in B, v \in A$  and  $\max_{x \in N(v) \cap B} \pi(x) > \pi(u)$

Let  $P_1$  denote the probability of situation 1 to occur,  $P_2$  that of situation 2 and  $P_3$  that of situation 3. Since all the three cases are mutually exclusive,  $\Pr[e \in E(G')] = P_1 + P_2 + P_3$ . It can be easily seen that  $P_1 = \Pr[u, v \in A] + \Pr[u, v \in B] = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ .  $P_2$  and  $P_3$  can be calculated as follows:

$$P_2 = \Pr[u \in A \wedge v \in B \wedge \max_{x \in N(u) \cap B} \pi(x) > \pi(v)]$$

Note that creating the random permutation and tossing the coins are two different experiments independent of each other. Moreover, the coin toss for each vertex is an experiment independent of all other coin tosses. Thus, the events  $u \in A, v \in B$  and  $\max_{x \in N(u) \cap B} \pi(x) > \pi(v)$  are all independent of each other. Therefore,

$$\begin{aligned}P_2 &= \Pr[u \in A] \times \Pr[v \in B] \times \\ &\quad \Pr[\max_{x \in N(u) \cap B} \pi(x) > \pi(v)]\end{aligned}$$

Now,  $\Pr[\max_{x \in N(u) \cap B} \pi(x) > \pi(v)] \leq \Pr[\max_{x \in N(u)} \pi(x) > \pi(v)] = p$  (say). Let  $X = \{v\} \cup N(u)$  and let  $\pi_X$  be the projection of  $\pi$  onto  $X$ . Then  $p$  is the probability that the condition  $\pi_X(v) \neq |X|$  is satisfied. Since  $\pi_X$  can be any permutation of  $|X| = d(u) + 1$  elements with equal probability  $\frac{1}{(d(u)+1)!}$  and the number of permutations which satisfy our condition is  $d(u)!d(u)$ ,  $p = \frac{d(u)!d(u)}{(d(u)+1)!} = \frac{d(u)}{d(u)+1}$ . Therefore,  $\Pr[\max_{x \in N(u) \cap B} \pi(x) > \pi(v)] \leq \frac{d(u)}{d(u)+1}$ . It can be easily seen that  $\Pr[u \in A] = \frac{1}{2}$  and  $\Pr[v \in B] = \frac{1}{2}$ . Thus,

$$P_2 \leq \frac{1}{2} \times \frac{1}{2} \times \frac{d(u)}{d(u) + 1} = \frac{1}{4} \left( \frac{d(u)}{d(u) + 1} \right)$$

Using similar arguments,

$$P_3 \leq \frac{1}{4} \left( \frac{d(v)}{d(v) + 1} \right)$$

Thus,

$$\begin{aligned}\Pr[e \in E(G')] &= P_1 + P_2 + P_3 \\ &\leq \frac{1}{2} + \frac{1}{4} \left( \frac{d(u)}{d(u)+1} + \frac{d(v)}{d(v)+1} \right)\end{aligned}$$

Hence the lemma.

**Theorem 2.** *Given a simple, undirected graph  $G$  on  $n$  vertices with maximum degree  $\Delta$ ,  $\text{cub}(G) \leq \lceil 4(\Delta + 1) \ln n \rceil$ .*

*Proof.* Let us invoke **RAND**( $G$ )  $k$  times so that we obtain  $k$  indifference supergraphs of  $G$  which we will call  $G'_1, G'_2, \dots, G'_k$ . Let  $G'' = G'_1 \cap G'_2 \cap \dots \cap G'_k$ . Obviously,  $G''$  is a supergraph of  $G$ . If  $G'' = G$ , then we have obtained  $k$  indifference graphs whose intersection gives  $G$ , which in turn means that  $\text{cub}(G) \leq k$ . The  $k$  indifference graphs can be seen as an indifference graph representation of  $G$ . We now estimate an upper bound for the value of  $k$  so that  $G'' = G$ .

Let  $(u, v) \notin E(G)$ .

$$\begin{aligned}\Pr[(u, v) \in E(G'')] &= \Pr \left[ \bigwedge_{1 \leq i \leq k} (u, v) \in E(G'_i) \right] \\ &\leq \left( \frac{2\Delta + 1}{2\Delta + 2} \right)^k \text{ (From lemma 1)}\end{aligned}$$

$$\begin{aligned}\Pr[G'' \neq G] &= \Pr \left[ \bigvee_{(u, v) \notin E(G)} (u, v) \in E(G'') \right] \\ &\leq \frac{n^2}{2} \left( \frac{2\Delta + 1}{2\Delta + 2} \right)^k \\ &= \frac{n^2}{2} \left( 1 - \frac{1}{2(\Delta + 1)} \right)^k \\ &\leq \frac{n^2}{2} \times e^{-\frac{k}{2(\Delta + 1)}}\end{aligned}$$

Choosing  $k = 4(\Delta + 1) \ln n$ , we get,

$$\Pr[G'' \neq G] \leq \frac{1}{2}$$

Therefore, if we invoke **RAND**  $\lceil 4(\Delta + 1) \ln n \rceil$  times, there is a non-zero probability that we obtain an indifference graph representation of  $G$ . Thus,  $\text{cub}(G) \leq \lceil 4(\Delta + 1) \ln n \rceil$ .

**Theorem 3.** *Given a graph  $G$  on  $n$  vertices with maximum degree  $\Delta$ . Let  $G_1, G_2, \dots, G_k$  be  $k$  indifference supergraphs of  $G$  generated by  $k$  invocations of **RAND**( $G$ ) and let  $G'' = G'_1 \cap G'_2 \cap \dots \cap G'_k$ . Then, for  $k \geq 6(\Delta + 1) \ln n$ ,  $G'' = G$  with high probability.*

*Proof.* Choosing  $k = 6(\Delta + 1) \ln n$  in the final step of proof of theorem 2, we get,

$$\Pr[G'' \neq G] \leq \frac{1}{2n}$$

Thus, if  $k \geq 6(\Delta + 1) \ln n$ ,  $G'' = G$  with high probability.

**Theorem 4.** *Given a graph  $G$  with  $n$  vertices,  $m$  edges and maximum degree  $\Delta$ , with high probability, its cube representation in  $\lceil 6(\Delta + 1) \ln n \rceil$  dimensions can be generated in  $O(\Delta(m + n) \ln n)$  time.*

*Proof.* We assume that a random permutation  $\pi$  on  $n$  vertices can be computed in  $O(n)$  time and that a random coin toss for each vertex takes only  $O(1)$  time. We take  $n$  steps to assign intervals to the  $n$  vertices. Suppose in a given step, we are attempting to assign an interval to vertex  $u$ . If  $u \in B$ , then we can assign the interval  $[n + \pi(u), 2n + \pi(u)]$  to it in constant time. If  $u \in A$ , We look at each neighbour of the vertex  $u$  in order to find out a neighbour  $v \in B$  such that  $\pi(v) = \max_{x \in N(u) \cap B} \pi(x)$  and assign the interval  $[\pi(v), n + \pi(v)]$  to  $u$ . It is obvious that determining this neighbour  $v$  will take just  $O(d(u))$  time. Since the number of edges in the graph  $m = \frac{1}{2} \sum_{u \in V} d(u)$ , one invocation of **RAND** needs only  $O(m + n)$  time. Since we need to invoke **RAND**  $O(\Delta \ln n)$  times (see the proof of Theorem 2), the overall algorithm that generates the cube representation in  $6(\Delta + 1) \ln n$  dimensions runs in  $O(\Delta(m + n) \ln n)$  time.

**Derandomization:** The above algorithm can be derandomized by adapting the techniques used in [4] to obtain a deterministic polynomial time algorithm **DET** with the same performance guarantee on the number of dimensions for the cube representation. Let  $t = \lceil 4(\Delta + 1) \ln n \rceil$ . Given  $G$ , **DET** selects  $t$  permutations  $\pi_1, \dots, \pi_t$  and  $t$  subsets  $A_1, \dots, A_t$  of  $V$  in such a way that the indifference graphs  $\{\mathcal{M}(G, \pi_i, A_i) \mid 1 \leq i \leq t\}$  form an indifference graph representation of  $G$ .

### 3 Improving to $O(\Delta \ln b)$ dimensions

In this section we show an algorithm **DETBAND** to construct the cube representation of  $G = (V, E)$  in  $O(\Delta \ln b)$  dimensions given a linear arrangement  $\mathcal{A}$  of  $V(G)$  with width  $b$ . The **DETBAND** algorithm internally invokes the **DET** algorithm (see the derandomization part of Section 2). Let the linear arrangement  $\mathcal{A}$  be  $v_1, \dots, v_n$ . For ease of presentation, assume that  $n$  is a multiple of  $b$ . Define a partition  $B_0, \dots, B_{k-1}$  of  $V$  where  $k = n/b$ , where  $B_j = \{v_{jb+1}, \dots, v_{jb+b}\}$ . Let  $H_i$  for  $0 \leq i \leq k-2$  be the induced subgraph of  $G$  on the vertex set  $B_i \cup B_{i+1}$ . Since for any  $i$ ,  $|V(H_i)| = 2b$ , we have  $\text{cub}(H_i) \leq \lceil 4(\Delta + 1) \ln(2b) \rceil = t$  (say). Let

$H_i^1, \dots, H_i^t$  be the indifference graph representation for  $H_i$  produced by **DET** when given  $H_i$  as the input. Let  $g_i^1, \dots, g_i^t$  be their corresponding unit interval representations.

Define, for  $0 \leq i \leq 2$ , the graph  $G_i$  with  $V(G_i) = V$  as the intersection of  $t$  indifference graphs  $I_{i,1}, \dots, I_{i,t}$ . Let  $f_{i,j}$  be the unit interval representation for  $I_{i,j}$ . For each vertex  $u \in V$ , define  $f_{i,j}(u)$  as follows:

If  $u \in V(H_s)$  such that  $s \in \{i, i+3, i+6, \dots\}$ , then define  $f_{i,j}(u) = g_s^j(u)$ . Otherwise, define  $f_{i,j}(u) = n$ .

The indifference graph  $I_0$  is constructed by assigning to each vertex in  $B_i$  the interval  $[in, (i+1)n]$ , for  $0 \leq i \leq k-1$ .

We prove that  $G = I_0 \cap G_0 \cap G_1 \cap G_2$  and thereby show that  $\text{cub}(G) \leq 12(\Delta + 1)\lceil \ln(2b) \rceil + 1$  or  $\text{cub}(G) = O(\Delta \ln b)$ .

**Definition 2.** Let  $I_1$  and  $I_2$  be two indifference graphs on disjoint sets of vertices  $V_1$  and  $V_2$  respectively. Let  $f_1$  and  $f_2$  be their corresponding unit interval representations. We say that a unit interval graph representation  $f : V_1 \cup V_2 \rightarrow \mathbb{R}$  of  $I_1 \cup I_2$  is a union of  $f_1$  and  $f_2$  if  $f(u) = f_1(u)$  if  $u \in V_1$  and  $f(u) = f_2(u)$  if  $u \in V_2$ .

Let  $t = \lceil 4(\Delta + 1) \ln(2b) \rceil$ .

### DET BAND

Input:  $G, \mathcal{A}$ .

Output: Representation of  $G$  using  $3t + 1$  indifference graphs.

**begin**

(The length of each interval is  $n$ )

Construct  $I_0$ : for each  $i$  and for each node  $v \in B_i$ ,  $f_0(v) = i \cdot n$ .

Construction of  $I_{i,j}$ ,  $0 \leq i \leq 2$  and  $1 \leq j \leq t$ :

Invoke **DET** on each induced subgraph in  $\mathcal{H} = \{H_{3r+i} : r = 0, 1, \dots\}$ .

Let  $H_k^1, \dots, H_k^t$  be the indifference graphs output by **DET** for  $H_k$ .

Let  $S = V - \bigcup_{H \in \mathcal{H}} V(H)$ .

Let  $f_S : S \rightarrow \mathbb{R}$  be defined as  $f_S(v) = n$  for all  $v \in S$ .

Define  $f_{i,j}$  as the union of  $f_S$  and the  $f$  functions corresponding to each graph in  $\{H_{3r+i}^j : r = 0, 1, 2, \dots\}$ .

**end**

**Theorem 5.** **DET BAND** constructs the cube representation of  $G$  in  $12(\Delta + 1)\lceil \ln(2b) \rceil + 1$  dimensions in polynomial time.

*Proof.* Let  $t = \lceil 4(\Delta + 1) \ln(2b) \rceil$ .

*Claim.*  $I_0$  is a supergraph of  $G$ .

*Proof.* Consider an edge  $(v_x, v_y) \in E(G)$  (assume  $x < y$ ). If  $B_m$  is the block containing  $v_x$ , then  $v_y$  is contained in either  $B_m$  or  $B_{m+1}$  since  $y - x \leq b$  and each block contains  $b$  vertices. Thus,  $f_0(v_x) = mn$  and  $f_0(v_y) = mn$  or  $mn + n$ . In either case, there is an overlap between  $f_0(v_x)$  and  $f_0(v_y)$  at the point  $mn$  and therefore,  $(v_x, v_y) \in E(I_0)$ .

*Claim.*  $I_{i,j}$ , for  $0 \leq i \leq 2$  and  $1 \leq j \leq t$ , is a supergraph of  $G$ .

*Proof.* Consider an edge  $(v_x, v_y) \in E(G)$  (assume  $x < y$ ). Let  $B_m$  be the block that contains  $v_x$ . From our earlier observation,  $v_y$  is either in  $B_m$  or in  $B_{m+1}$ .

If  $v_x, v_y \in V(H_p)$ , where  $p = 3r + i$  for some  $r \geq 0$ , then by definition of  $f_{i,j}$ ,  $f_{i,j}(v_x)$  and  $f_{i,j}(v_y)$  correspond to the intervals assigned to them in the interval representation of the indifference graph  $H_p^j$ . Since  $(v_x, v_y) \in E(H_p)$  and  $H_p \subseteq H_p^j$ , the intervals  $f_{i,j}(v_x)$  and  $f_{i,j}(v_y)$  overlap and therefore  $(v_x, v_y) \in E(I_{i,j})$ .

Now, if  $m = 3r + i$ , for some  $r \geq 0$ , then  $v_x, v_y \in H_m$ . Therefore, as detailed in the previous paragraph, it follows that  $(v_x, v_y) \in E(I_{i,j})$ .

If  $m = 3r + i + 1$ , for some  $r \geq 0$ , then either  $v_x, v_y \in V(H_{m-1})$  or  $v_x \in V(H_{m-1})$  and  $v_y \in S$ . In the first case, the earlier argument can be applied again to obtain the result that  $(v_x, v_y) \in E(I_{i,j})$ . Now, if  $v_x \in V(H_{m-1})$  and  $v_y \in S$ , we have  $m - 1 = 3r + i$  and therefore by definition of  $f_{i,j}$ ,  $f_{i,j}(v_x)$  is the interval mapped to  $v_x$  in the interval representation of the indifference graph  $H_{m-1}^j$ . From the construction of **DET**, it is clear that  $0 \leq f_{i,j}(v_x) \leq 2n$  (see the derandomization part of section 2). Also, we have  $f_{i,j}(v_y) = f_S(v_y) = n$ . It can be seen that  $|f_{i,j}(v_x) - f_{i,j}(v_y)| \leq n$  and therefore  $(v_x, v_y) \in E(I_{i,j})$ .

Similarly, if  $m = 3r + i + 2$ , for some  $r \geq 0$ , then  $v_x \in S$  and  $v_y$  is contained either in  $S$  or in  $V(H_{m+1})$ . It can be shown using arguments similar to the ones used in the preceding paragraph that  $(v_x, v_y) \in E(I_{i,j})$ .

This completes the proof that  $G \subseteq I_{i,j}$ , for  $0 \leq i \leq 2$ ,  $1 \leq j \leq t$ .

*Claim.* The indifference graphs  $I_{i,j}$ , for  $0 \leq i \leq 2$  and  $1 \leq j \leq t$ , along with  $I_0$  constitute a valid indifference graph representation of  $G$ .

*Proof.* We have to show that given any edge  $(v_x, v_y) \notin E(G)$ , there is at least one graph among the  $3t + 1$  indifference graphs generated by **DETBAND** that does not contain that edge.

Assume that  $x < y$ . Let  $B_m$  and  $B_l$  be the blocks containing  $v_x$  and  $v_y$  respectively. If  $l - m > 1$  then  $f_0(v_y) - f_0(v_x) = (l - m)n > n$ . Therefore,  $(v_x, v_y) \notin E(I_0)$ . Now we consider the case when  $l - m \leq 1$ . Consider the set of indifference graphs  $\mathcal{I} = \{H_m^j \mid 1 \leq j \leq t\}$  that is generated by **DET** when given  $H_m$  as input. We know that  $(v_x, v_y) \notin E(H_m)$  because  $H_m$  is an induced subgraph of  $G$  containing the vertices  $v_x$  and  $v_y$ . Since  $\mathcal{I}$  is a valid indifference graph representation of  $H_m$ , at least one of the graphs in  $\mathcal{I}$ , say  $H_m^p$ , should not contain the edge  $(v_x, v_y)$ . Let  $g$  be the unit interval representation function corresponding to  $H_m^p$  as output by **DET**. Since  $(v_x, v_y) \notin E(H_m^p)$ ,  $|g(v_x) - g(v_y)| > n$ . Let  $i = m \bmod 3$ . Thus,  $m = 3r + i$ , for some  $r \geq 0$ . Now,

since  $f_{i,p}$  is defined as the union of the unit interval representation functions of all the graphs in the set  $\{H_{3r+i}^p : r = 0, 1, 2, \dots\}$  which contains  $H_m^p$ ,  $f_{i,p}(v_x) = g(v_x)$  and  $f_{i,p}(v_y) = g(v_y)$  which implies that  $|f_{i,p}(v_x) - f_{i,p}(v_y)| > n$ . Therefore,  $(v_x, v_y) \notin E(I_{i,p})$ .

Thus, **DETBAND** generates a valid indifference graph representation of  $G$  using  $3t + 1 \leq 12(\Delta + 1)\lceil \ln(2b) \rceil$  indifference graphs. Since **DET** runs in polynomial time and there are only polynomial number of invocations of **DET**, the procedure **DETBAND** runs in polynomial time.

**Tight example:** Consider the case when  $G$  is a complete binary tree of height  $d = \log n$ . Using the results shown in [6], we can see that  $\text{cub}(G) \geq \frac{d}{\log 2d} = \frac{\log n}{c_1 + \log \log n}$  where  $c_1$  is a constant. Therefore,  $\text{cub}(G) = \Omega(\frac{\log n}{\log \log n})$ . From theorem 2,  $\text{cub}(G) \leq 4(\Delta + 1) \ln n = 16 \ln n = c_2 \log n$ , where  $c_2$  is a constant. Therefore, the upper bound provided by theorem 2 is tight up to a factor of  $O(\log \log n)$ . Since the bandwidth of the complete binary tree on  $n$  vertices is  $\Theta(\frac{n}{\log n})$  as shown in [12], the  $O(\Delta \ln b)$  bound on cubicity is also tight up to a factor of  $O(\log \log n)$ .

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